PORTFOLIO OPTIMIZATION WITH SHORTAGE FUNCTION AND HIGHER ORDER MOMENTS: AN APPLICATION IN ISE-30

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Abstract: In this paper, we propose an optimal portfolio selection with nonparametric criteria using the mean-variance-skewness-kurtosis framework. By construction of a polynomial goal programming (PGP), we analyze multiple competing portfolio allocation objectives such as maximizing expected portfolio return and skewness, minimizing risk and kurtosis simultaneously and investor’s preferences over incorporation of higher moments. The works of Jurczenko et al. (2006) and Briec et al. (2007) introduced a shortage function which is defined that search for possible increases in expected portfolio return and skewness and decreases in variance and kurtosis. The empirical application investigates the existence of an optimal solution with higher moments in Turkish market, how the incorporation of different investors’ preferences and the four moment of stock return.

Keywords: portfolio selection, global optimization, shortage function, mean-variance-skewness-kurtosis portfolio, multi-objective optimization.

1. Introduction

In the modern portfolio theory based on normal distribution, the model with mean-variance (MV) criterion proposed by Markowitz (1952) has been playing a significant role and has acquired common acceptance as a tool for portfolio optimization. After Markowitz (1952) study, many studies on portfolio allocation and performance measure have been made based on only first two moment of return distribution. However, since Mandelbrot (1963), many economists and researchers have indicated that rates of return of a portfolio or assets very often are not normally distribution. Many academic studies like Konno and Suzuki (1995), and Anson (2002) argued that two moment performance measures for portfolio optimization may not adequate if asset or portfolio returns distributions are asymmetrical and investor’s utility functions are higher order than quadratic. Popova et al. (2003), and Davies et al. (2005) implemented these issues by using higher moment analysis.

Lai (1991) applied polynomial goal programming (PGP) to find out the portfolio allocation with skewness. In the presence of skewness, selecting a portfolio is a trade-off between competing and conflicting objectives. Lai (1991) indicated that investor’s preferences can be incorporated into PGP from which a portfolio allocation with skewness is determined. Davies et al. (2005) and Berényi (2005) used PGP approach to determine the set of the mean-variance-skewness-kurtosis (MVK) efficient funds of hedge funds. Athayde and Flôres (2003), and Jurczenko and Maillet (2006) look for the analytical solution characterizing the minimum variance frontier in the MVSK space, whereby the objective is to minimize the variance for a given mean, skewness and kurtosis.

Briec et al. (2004) applied the shortage function which was first introduced by Luenberger (1995) in production theory. It is a distance function that looks simultaneously for reduction in inputs and expansion in outputs. It has been subsequently used by Morey and Morey (1999) and Briec et al. (2004) for gauging the performance of funds in the MV framework and more recently by Briec et al. (2007) for solving portfolio selection problems involving significant degree of skewness.

* Any opinions expressed in this article are those of the author and may not necessarily reflect the opinions of Turkish Derivatives Exchange (Vadeli İşlem ve Opsiyon Borsası A.S).
In the MVSK framework, we find out the problem of multiple conflicting and competing portfolio objectives such as maximizing expected return and skewness and minimizing risk and kurtosis simultaneously, by construction of a PGP model into which the specific investor’s personal preferences are incorporated under no-short selling. The empirical application on a small sample of stocks in Istanbul Stock Exchange (ISE) investigates the existence of an optimal solution with shortage function and higher moments in Turkish market, how the incorporation of different investors’ preferences and the four moment of stock return and no-short selling.

2. Portfolio Optimization in Mean – Variance – Skewness – Kurtosis Framework

Different approaches have been proposed in the literature for portfolio optimization problem under higher moments. Many researchers used polynomial goal programming (PGP) to solve the portfolio selection with skewness and/or kurtosis. To solve this problem, one has to specify the investor’s subjective judgments and relative preferences on objectives. Some researchers provided a quasi-analytic solution to the efficient portfolio problem by defining moments as tensors and then solving the optimization problem.

The PGP was first introduced by Tayi and Leonard (1988) to facilitate bank balance sheet management with competing and conflicting objectives. It has subsequently been used for solving portfolio selection in MVS framework by Lai (1991), Chunhachinda et al. (1997), Wang and Xia (2002), Prakash et al. (2003). More recently, techniques have been developed to solve portfolio allocation problem. Jurczenko et al. (2006), Lai et al. (2006), Briec and Kerstens (2009), Kleniati and Rustem (2009) and Harvey et al. (2004) applied PGP approach to the portfolio allocation with skewness and kurtosis.

Let \( w_p \) denote respectively the \((N \times 1)\) vector of weights and \( E \) the expected returns of the \( N \) risky assets in the portfolio \( p \); \( \Sigma \) be the non-singular \((N \times N)\) variance-covariance matrix of the risky assets; and \( \Omega \) and \( \Gamma \) represent respectively the \((N \times N^2)\) skewness-coskewness matrix and the \((N \times N^3)\) kurtosis-cokurtosis matrix of the \( N \) risky asset returns (see, Briec et al., 2007, Jurczenko et al., 2006, Athayde and Flôres, 2004). The set of the feasible portfolios \( \tilde{F}_p \) can be expressed as follows:

\[
\tilde{F}_p = \{w_p \in IR^N : w'_p = 1 \text{ and } w'_p \geq 0\}
\]

where \( w'_p \) is the \((1 \times N)\) transposed vector of the investor’s holdings of risky assets and \( 1 \) the \((N \times 1)\) unitary vector. The mean, variance, skewness and kurtosis of the return of a given portfolio \( p \) belonging to \( \tilde{F}_p \) are respectively given by (2.1) with \( \forall (i, j, k, l) \in [1, ..., N]^4 \) where \( w_{pi}, R_i, (\sigma_{ij})_{i,j}, (s_{ijk})_{i,j,k} \) and \((\kappa_{ijkl})_{i,j,k,l}\) represent, respectively, the weight of the assets \( i \) in portfolio \( p \), the return on the asset \( i \), the covariance between the returns of asset \( i \) and \( j \), the coskewness between the return of asset \( i, j \) and \( k \) and the cokurtosis between the returns of asset \( i, j, k \) and \( l \) (see Jurczenko et al., 2006).

The following disposal representation, denoted \( \otimes m_p \) in (2.2), of the set of the feasible portfolios in the MVSK space, where \( m_p \) is the \((4 \times 1)\) vector of the first four moments of the portfolio return \( p \). This

\[
\begin{align*}
\mathbb{E}(R_p) &= \mathbb{E}\left[ \sum_{i=1}^{N} w_{pi} R_i \right] = w'_p E \\
\sigma^2(R_p) &= \mathbb{E}\left\{ [R_p - \mathbb{E}(R_p)]^2 \right\} = \sum_{i=1}^{N} \sum_{j=1}^{N} w_{pi} w_{pj} \sigma_{ij} = w'_p \Sigma w_p \\
s^3(R_p) &= \mathbb{E}\left\{ [R_p - \mathbb{E}(R_p)]^3 \right\} = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} w_{pi} w_{pj} w_{pk} s_{ijk} = w'_p \Omega(w_p \otimes w_p) \\
\kappa^4(R_p) &= \mathbb{E}\left\{ [R_p - \mathbb{E}(R_p)]^4 \right\} = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{l=1}^{N} w_{pi} w_{pj} w_{pk} w_{pl} \kappa_{ijkl} = w'_p \Gamma (w_p \otimes w_p \otimes w_p)
\end{align*}
\]

where the sign \( \otimes \) stands for the Kronecker symbol product.
disposal representation is necessary to ensure the convexity of the feasible portfolio set in the MVSK space. The four-moment (weakly) efficient portfolio frontier is the set of all the mean, variance, skewness, kurtosis quadruplets that are not strictly dominated in the four-dimensional space (see Briec et al. 2004 and 2007, and Jurczenko et al., 2006).

\[ \mathcal{D}(\mathcal{M}_p) = \{ m_p : w_p \in \mathcal{D}(\mathcal{M}_p) \} + [IR_+ \times IR_- \times IR_+ \times IR_-] \], \quad \mathcal{M}_p = [\kappa^4(R_p) s^3(R_p) \sigma^2(R_p) \mu(R_p)]^T \quad (2.2)

3. The Shortage Function and the Four Moment Efficient Frontier

In production theory, the shortage function measures the distance between some point of the production possibility set and the efficient production frontier (see Luenberger, 1995). The properties of the set of portfolio return moments on which the shortage function is defined have been discussed by Briec et al. (2007) for the MVS portfolio selection framework and Jurczenko et al. (2006) for the MVSK portfolio selection frameworks.

The shortage function associated to feasible portfolio \( p \) with reference to the direction vector \( g \) in the four moments space is the real-valued function \( S_g(w) \) defined as (see Jurczenko et al., 2006):

\[ S_g(w) = \sup\{ \delta : m_p + \delta g \in \mathcal{D}(\mathcal{M}_p) | g \in [IR_+ \times IR_- \times IR_+ \times IR_-] \} \]

The disposal representation of the feasible portfolio set can be used to derive the lower bound of the true unknown four moment efficient frontier through the computation of the associated portfolio shortage function. Let consider a specific portfolio \( w_k \) from a sample of \( p \) portfolios – or assets – \( (w_p)_p=1,...,p \) whose performance need to be evaluated in the four-moment space. The shortage function for this portfolio is then computed by solving the following quartic optimization program in (3.1) where \( w_{p*} \) represent the \( (N \times 1) \) efficient portfolio weight vector that maximize the expected return, risk, skewness and kurtosis improvement with respect to the ones of the evaluated portfolio in the direction vector \( g \). Using the vectorial notation of portfolio return higher moments and using the first four moments of the evaluated portfolio \( k \) in the expression of the direction vector \( g \), the nonparametric portfolio optimization program can then be restated as (see Jurczenko et al., 2006),

\[ w_{p*} = \text{Arg}\{ \text{Max}_w \delta \} \]

\[ \begin{align*}
\mathbb{E}(R_k) + \delta \mathbb{E}(R_k) & \leq w_p E \\
\sigma^2(R_k) - \delta \sigma^2(R_k) & \geq w_p \Sigma w_p \\
s^3(R_k) + \delta s^3(R_k) & \leq w_p \Omega(w_p \otimes w_p) \\
\kappa^4(R_k) - \delta \kappa^4(R_k) & \geq w_p \Gamma(w_p \otimes w_p \otimes w_p) \\
w_{p*1} & = 1 \\
w_{p*1} & \geq 0 \\
\end{align*} \quad (3.1)

Thus, gauging the performance of a sample of \( w_p \) portfolios requires computing one mathematical program for each portfolio in turn to determine its position with respect to the boundary of the choice set. Combinations of moments of the portfolios in the sample constituting the portfolio frontier are situated on the right-hand-side of (3.1). The evaluated portfolio is represented on the left-hand-side of (3.1). Maximizing \( \delta \) attempts to augment its odd moments and reduce its even moments in the direction of vector \( g \). If \( \delta = 0 \), then the evaluated portfolio is efficient and on the boundary of the portfolio frontier. If \( \delta \neq 0 \), there are combinations of portfolios that yield higher odd moments and lower even moments. Hence, the evaluated portfolio is situated below the boundary and inefficient.

In order to allow the investor exhibiting asymmetric preference towards the mean, variance, skewness and kurtosis of return, parameters \( \lambda_1, \lambda_2, \lambda_3 \) and \( \lambda_4 \) to indicate the relative preference for mean, variance,
skewness and kurtosis, respectively and the investor’s preferences parameters’ values can be 0, 1, 2, 3. We can reformulate (3.1) to the minimization optimization algorithm of the PGP model and our modified objective function with given investor’s relative preference for mean, variance, skewness and kurtosis of return can be represented as,

\[
\begin{align*}
    w_{p^*} = & \text{Arg}\{\text{Max} \ \tilde{\sigma}\} \\
\begin{cases}
    \lambda_1 \delta \leq \left( \frac{-\mathbb{E}(R_k) + w_p^* E}{g_E} \right) * \lambda_1 \\
    \lambda_2 \delta \leq \left( \frac{\sigma^2(R_k) - w_p^* \Sigma \rho_k}{g_v} \right) * \lambda_2 \\
    \lambda_3 \delta \leq \left( \frac{-s^3(R_k) + w_p^* \Omega (w_p^* \otimes w_p^*)}{g_v} \right) * \lambda_3 \\
    \lambda_4 \delta \leq \left( \frac{\kappa^4(R_k) - w_p^* \Gamma (w_p^* \otimes w_p^* \otimes w_p^*)}{g_k} \right) * \lambda_4 \\
\end{cases}
\end{align*}
\]

(3.2)

\[
g = [-\kappa^4(R_k)s^3(R_k) - \sigma^2(R_k) \mathbb{E}(R_k)]', \quad \lambda = [\lambda_1 \lambda_2 \lambda_3 \lambda_4]', \quad \varpi = [\lambda_1 \lambda_2 \lambda_3 \lambda_4]' \delta
\]

4. Empirical Results

In this study, we present a Turkish Stock Market example to illustrate the algorithm of portfolio selection in MVSK space. The implementation serves three purposes: a) to show how portfolio selection will vary for different investor’s preferences structure, b) to compare four-moment world with Markowitz’s two-moment world, c) to show three-dimensional representation of the non-convex MVSK efficient frontier. We use 8 securities in Istanbul Stock Exchange (ISE) for the implementation. Our data set includes historical stock prices obtained from [http://www.platodata.com.tr/](http://www.platodata.com.tr/). The stocks in ISE considered the permanent stocks in ISE-30 Index and the stock’s price data set covers the period from January 2, 2009 to December 31, 2009 on a daily basis. We use logarithmic returns to convert the historical prices into assets returns. We assume that the investor does not have access to a riskless asset, implying that the portfolio weights must sum to one. In addition we use the standard portfolio selection assumptions, which are perfect divisibility of the invested strategies, no taxes, no transaction costs, and perfect markets. Short selling is not permitted that is the portfolio weights must be positive and must sum to one. Firstly, the distribution properties of the returns of 8 stocks is calculated and presented in Table 1. In accordance with the third section, by solving each objective with linear and nonlinear programming techniques, we can calculate aspired level of the optimal solution of individual objective, as shown in Table 2. After calculating the aspired level, we can solve (3.2) with the PGP approach. Given investor’s preference of \( \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1 \) which means the weight for four moment are equivalent, the global optimal solution set of PGP portfolio optimization with the shortage function are illustrated in Table 3. In addition, the three-dimensional representation of the non-convex MVSK efficient frontier lines of the PGP optimal model with the shortage function is shown in Fig. 1 and Fig. 2. However, an important factor, investor’s preferences, is not unchangeable in the process of investment. In addition investor’s preferences often affect the investment strategy (Lai et al., 2006). For this purpose, different levels of preference are examined. Table 4 presents that the optimization of the first four moments of portfolio consisting of 8 stocks and several different investors’ preference scenarios including the Markowitz benchmark case \( \lambda_1 = \lambda_2 = 1, \lambda_3 = \lambda_4 = 0 \).
Table 1. The Distribution properties of assets return

<table>
<thead>
<tr>
<th>Stock</th>
<th>Mean</th>
<th>Variance</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>Stock</th>
<th>Mean</th>
<th>Variance</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>GARAN</td>
<td>0.003570</td>
<td>0.000841</td>
<td>-0.141290</td>
<td>4.202570</td>
<td>YAKBN</td>
<td>0.004950</td>
<td>0.000811</td>
<td>0.078730</td>
<td>3.032760</td>
</tr>
<tr>
<td>ISCTR</td>
<td>0.002210</td>
<td>0.000653</td>
<td>0.073310</td>
<td>3.428970</td>
<td>YKBNK</td>
<td>0.001800</td>
<td>0.000671</td>
<td>-0.120150</td>
<td>3.551160</td>
</tr>
<tr>
<td>AKBK</td>
<td>0.002730</td>
<td>0.000945</td>
<td>0.438460</td>
<td>4.243480</td>
<td>SAHOL</td>
<td>0.002420</td>
<td>0.000735</td>
<td>-0.179770</td>
<td>3.818950</td>
</tr>
<tr>
<td>TCELL</td>
<td>0.001040</td>
<td>0.000398</td>
<td>0.107200</td>
<td>3.423680</td>
<td>KCHOL</td>
<td>0.002830</td>
<td>0.000571</td>
<td>0.051280</td>
<td>2.935660</td>
</tr>
</tbody>
</table>

Table 2. Each objective’s optimal optimization scores

<table>
<thead>
<tr>
<th>Objectives</th>
<th>Mean*</th>
<th>Variance*</th>
<th>Skewness*</th>
<th>Kurtosis*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Optimal Score</td>
<td>0.00495</td>
<td>0.000398</td>
<td>0.43846</td>
<td>2.9270</td>
</tr>
</tbody>
</table>

Table 3. Optimal solution set of PGP portfolio optimization scores

<table>
<thead>
<tr>
<th>Objectives</th>
<th>Mean*</th>
<th>Variance*</th>
<th>Skewness*</th>
<th>Kurtosis*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Optimal Score</td>
<td>0.0027</td>
<td>0.00095</td>
<td>0.43846</td>
<td>4.2435</td>
</tr>
</tbody>
</table>

Table 4. Trade-off between the first four moments with different investors’ preferences and optimal PGP score

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Mean</th>
<th>Variance</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.00273214</td>
<td>0.00094941</td>
<td>0.43846444</td>
<td>4.24348260</td>
</tr>
<tr>
<td>B</td>
<td>0.00273214</td>
<td>0.00094941</td>
<td>0.43846444</td>
<td>4.24348260</td>
</tr>
<tr>
<td>C</td>
<td>0.00331042</td>
<td>0.00078640</td>
<td>0.29327845</td>
<td>3.98534989</td>
</tr>
<tr>
<td>D</td>
<td>0.00273214</td>
<td>0.00094941</td>
<td>0.43846444</td>
<td>4.24348260</td>
</tr>
<tr>
<td>E</td>
<td>0.00494922</td>
<td>0.0081076</td>
<td>0.07873254</td>
<td>3.03275686</td>
</tr>
<tr>
<td>F</td>
<td>0.00104060</td>
<td>0.00039803</td>
<td>0.10720418</td>
<td>3.42368020</td>
</tr>
<tr>
<td>G</td>
<td>0.00273214</td>
<td>0.00094941</td>
<td>0.43846444</td>
<td>4.24348269</td>
</tr>
<tr>
<td>H</td>
<td>0.00285083</td>
<td>0.00055730</td>
<td>0.04652753</td>
<td>2.92698587</td>
</tr>
</tbody>
</table>

In Portfolio B, the four moment and set equal to those of Portfolio A. Investors higher preference for expected return and skewness in Portfolio C leads to a higher expected return than that in Portfolio A, resulting in a lower skewness than Portfolio A. Thus, as the investor’s preference for expected returns increases, he must settle for lower skewness. Investor’s higher preference for variance and kurtosis in Portfolio D equals to Portfolio A. In Portfolio E-H, in turn, the first moment, the second moment, the third moment and the fourth moment are optimized.

Fig. 1. M-V-S Efficient Frontier

Fig. 2. M-V-K Efficient Frontier
4. Conclusion

In this study, we have shown a general method for the set of efficient portfolios in the four moment space, using a shortage function with incorporated investor’s preferences framework and a development of a robust non-parametric multi-moment efficient frontier. The empirical application has provided a three-dimensional representation of the primal non-convex four-moment efficient portfolio frontier. The empirical analysis results indicate that the PGP approach with shortage function which is modified by investor’s preference can figure out portfolio selection problem with multiple objectives and can find different optimal portfolio under different investment strategy. The empirical illustration also shows that the trade-off between the first four moments with different investors’ preferences affects portfolio set and optimum moment score. For example, the higher investor’s preference for first and third moment, the higher expected return and the less variance, skewness and kurtosis. After Markowitz’s approach, many researches indicated that the shape of return distribution has heavy tails, and it is skewed and straight. The shape of the distribution should be determined much better in the four moment approach, so the method is more preferable than the Markowitz's approach. Our work results also show that the incorporation of investor’s preference, the impacts of coskewness and cokurtosis of assets’ return and global optimization produce higher return and less variance than the Markowitz’s approach.

References


