ON ONE APPROACH TO STUDY OF DEPARTURE PROCESS IN BATCH ARRIVAL QUEUES WITH MULTIPLE VACATIONS

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Abstract. A batch arrival single-server queueing system of a general type is considered. At the end of each busy period the server begins a multiple vacation period when the service process is stopped. A new approach to the study of departure process in such a system is proposed. Using the formula of total probability and renewal theory we direct the analysis to that in the system without vacations. An explicit representation for the double transform of probability function of departure process is derived. The formula is written down by means of transforms of distributions describing the arrival and service processes, vacation times and components of a certain factorization identity of Wiener-Hopf type connected with these distributions.

Keywords: Batch arrival queueing system, departure process, factorization of Wiener-Hopf type, multiple vacation, transient state.

1. Introduction

Considering systems with a server that is temporary unavailable becomes more and more important in the queueing theory. Many practical telecommunication, network and manufacturing problems lead to such systems. In the article we consider a general queueing system with batch arrivals, unlimited queue, multiple server vacations and exhaustive service. The work of such a system can be described as follows. The system is empty before initializing and at time \( t=0 \) the first group of customers arrives (we call such initial conditions as "standard" regime of the work). Every time when the system becomes empty (at the end of successive busy periods), the server begins a multiple vacation period during that the service process is blocked. Each multiple vacation period consists of a certain (finite) number successive single vacation times which are independent and identically distributed (i.i.d.) random variables. The server takes on successive single vacation times as far as there will be at least one customer waiting in the queue after one of them.

Majority of published results for models with multiple vacations relates to systems of M/G/1 type. One can find the in-depth review of results concerning queues with vacations up to 1986 in Doshi (1986). To the various vacation models is devoted a monograph Takagi (1991). The transient state of the M/G/1 queue with multiple vacations is investigated in Tang (1997). In Tang (2007) the mean of departures occurring before indicated moment \( t \) is derived for the system of \( M^N/\text{G}/1 \) type with multiple vacations.

In the paper we propose a new approach to the study of departure process in multiple vacation queues with exhaustive service. The main idea of this method is to direct the analysis of a certain auxiliary system with vacations to the analysis of the corresponding system without vacations under two different initial conditions and, next, to apply results derived for systems without vacations. Results for a general case we obtain applying the renewal-theory approach.

Main characteristics of batch arrival queueing systems without vacations were investigated in author's papers Bratiychuk; Kempa (2003 and 2004) and Kempa (2004 and 2008a). The formulae were written in innovative forms: using transforms of distributions characterizing the arrival and service processes and by means of components of a certain factorization identity of Wiener-Hopf type but without introducing any additional random walk.

Before we will state results we describe the model more precisely and give some auxiliary facts and notations. Assume that interarrival times are i.i.d. random variables with a distribution function (d.f.) \( F_1(\cdot) \); service times are i.i.d. random variables with a d.f. \( F_2(\cdot) \) and sizes of successive batches have the
same distribution given by a sequence \( \{ p_k \} \), \( k \geq 1 \). The work of the system can be observed on successive vacation cycles \( C_i, i = 0, 1, 2, \ldots \), which begin every time if the system becomes free of customers. Denoting by \( \tau_i, i = 0, 1, 2, \ldots \) successive busy periods of the system and by \( V_i, i = 1, 2, \ldots \) successive vacation periods we can write
\[
C_0 = \tau_0, \quad C_i = V_i + \tau_i, \quad i = 1, 2, \ldots .
\]

Moreover
\[
V_i = \sum_{k=1}^{r_i} u_{ik},
\]

where \( u_{ik} \) are i.i.d. single vacation times with a d.f. \( G(\cdot) \) and \( r_i \) denotes the number of single vacation times in the \( i \)th vacation period \( V_i \). We assume that vacation times are independent on the arrival process of customers.

Besides let for \( \text{Re}(s) \leq 0 \) and \( |\theta| \leq 1 \)
\[
f_i(s) = \int_0^{\infty} e^{-sx} dF_i(x), \quad i = 1, 2, \quad p(\theta) = \sum_{k=1}^{\infty} p_k \theta^k.
\]

By \( F_i(s) \) we denote the \( i \)-fold convolution of the d.f. \( F_i(\cdot) \) with itself and by \( \{ p_k \} \) - the \( i \)-fold convolution of the sequence \( \{ p_k \} \) with itself. We will also use the symbol \( I \{ \Lambda \} \) as the indicator of random event \( \Lambda \). Let \( h(t) \) stands for the number of customers completely served in the interval \([0, t]\).

Let us call an "ordinary" one the system without vacations corresponding to this with multiple vacations. According to the work of the "ordinary" system we introduce two initial conditions. Let
1. \( \mathbf{P}_n \{ \cdot \} \) and \( \mathbf{E}_n \{ \cdot \} \) stand respectively for the probability and mean under the condition that there are \( n \) customers present in the system just after the initial moment i.e. at time \( t=+0; \)
2. \( \mathbf{P}_\text{std} \{ \cdot \} \) and \( \mathbf{E}_\text{std} \{ \cdot \} \) stand respectively for the probability and mean under the condition that the system works in the "standard" regime i.e. it is empty before initializing and the first arrival occurs exactly at time \( t=0 \).

One can find the following result in Bratiychuk; Kempa (2003).

**Lemma 1.** For \( \text{Re}(s) \in [0, \lambda] \), \( |\theta| \leq 1 \) the following canonical factorization identity of Wiener-Hopf type holds true
\[
1 - f_1(s)p(\theta f_2(\lambda - s)) = f_+(\theta, \lambda, s)f_-(\theta, \lambda, s).
\]

The components \( f_\pm(\theta, \lambda, s) \) have the following properties
\[
\frac{1}{f_\pm(\theta, \lambda, s)} = 1 + \int_0^{\pm \infty} e^{-sx} dP_\pm(\theta, \lambda, x),
\]

where \( P_\pm(\theta, \lambda, x) \) are zero on a half-axis \( \mp x > 0 \) respectively and have a bounded variation for any \( \lambda > 0 \) and \( |\theta| \leq 1 \). Additionally let
\[
P_+(0)(\theta, \lambda, x) = I \{ x > 0 \} + P_+(\theta, \lambda, x).
\]

**Some auxiliary results for the system without vacations**

Let us consider the batch arrival queueing system without vacations corresponding to that with multiple vacations (the "ordinary" system). We consider the functioning of the "ordinary" system on its first busy period \( \tilde{\tau}_1 \). Let us denote by \( \tilde{h}(t) \) the number of customers being completely served during \([0, t]\) in the "ordinary" system and let for \( |\theta| \leq 1 \), \( \lambda > 0 \)
\[
\hat{D}_+(\theta, \lambda) = \sum_{m=0}^{\infty} \theta^m \int_0^{\infty} e^{-\lambda t} P_+(\hat{h}(t) = m, \ t \in \tilde{\tau}_1) \, dt,
\]

where \( \ast \) stands for "n" or "std". The following theorem is proved in Kempa (2008b) (a similar representation was obtained for \( \hat{D}_n(\theta, \lambda) \)).
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Theorem 1. For any $\lambda > 0$ and $|\theta| \leq 1$ the following equality is satisfied
\[
\hat{D}_{\text{std}}(\theta, \lambda) = \frac{\theta(1 - f_2(\lambda))}{\lambda(1 - \theta f_2(\lambda))} \left[ 1 - p(\theta f_2(\lambda)) + f_+(\theta, \lambda, 0) \times \right.
\]
\[
\times \int \frac{e^{-\lambda y}}{n} \int_{-\infty}^{y} F_1(y - v) dP^{(0)}_+(\theta, \lambda, v) \sum_{n=1}^{\infty} p_n \theta^n dF^{\ast n}(y) \right],
\] (7)

In Bratiychuk; Kempa (2003) the following result can be found.

Lemma 2. For any $\lambda > 0$ we have
\[
E_{\text{std}}\{e^{-\lambda \hat{r}_1}\} = 1 - f_+(1, \lambda, 0).
\] (8)

Similarly, in Bratiychuk; Kempa (2004) the following formula is derived.

Lemma 3. For $\lambda > 0$ and $n > 0$ we have
\[
E_{n}\{e^{-\lambda \hat{r}_1}\} = f_2^n(\lambda) - f_+(1, \lambda, 0) \times
\]
\[
\times \int \frac{e^{-\lambda y}}{y} \int_{-\infty}^{y} F_1(y - v) dP^{(0)}_+(1, \lambda, v) dF^{\ast n}_2(y).
\] (9)

2. Results

First, let us consider an “auxiliary” queueing system with multiple vacations. Such a system starts working at time $t = 0$ with no customers present and the first vacation cycle $C_1$ begins immediately. We will denote all probabilities for such a system by $P_0(\cdot)$. Using the formula of total probability we can express the p.f. (probability function) of departure process $h(t)$ in the system with multiple vacations on the first vacation cycle $C_1$ by means of the p.f. of $\tilde{h}(t)$ in the “ordinary” system. Indeed, we obtain
\[
P_0\{h(t) = m, t \in C_1\} =
\]
\[
\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{n=k+1}^{\infty} p_0^{(k+1)*} \int_{0}^{t} \int_{0}^{y} \left[ F_1^k(z + y - x) - F_1^{(k+1)*}(z + y - x) \right] \times
\]
\[
\times P_n\{\tilde{h}(t - z - y) = m, t - z - y \in \tilde{r}_1\} dG(z) dG^{\ast n}(y) dF_1(x)
\]
\[
+ I\{m = 0\} \sum_{i=0}^{\infty} \int_{0}^{t} \int_{0}^{y} (1 - G(t - y)) dG^{\ast n}(y) dF_1(x) + (1 - F_1(t)) \right].
\] (10)

Let us briefly comment equality (10). The first summand on the right side presents the situation in that the first arrival occurs before $t$ and the first multiple vacation period also ends before $t$. The second summand is connected with the situation in that either the first arrival occurs after $t$ or the first vacation period ends after $t$. Denoting
\[
D_0(\theta, \lambda) = \sum_{m=0}^{\infty} \theta^m \int_{0}^{\infty} e^{-\lambda y} P_0\{h(t) = m, t \in C_1\} dt,
\]
we obtain from (10)
\[
D_0(\theta, \lambda) = \sum_{i=0}^{\infty} \int_{0}^{\infty} e^{-\lambda y} \int_{0}^{t} K_i(t, x) dF_1(x) dt + \frac{1 - f_1(\lambda)}{\lambda}
\]
\[
+ \sum_{m=0}^{\infty} \theta^m \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{n=k+1}^{\infty} p_0^{(k+1)*} \hat{D}_n(\theta, \lambda) \times
\]
\[
\times \int_{0}^{\infty} e^{-\lambda y} dG^{\ast n}(y) \int_{0}^{\infty} e^{-\lambda z} T(y, z, k) dG(z),
\] (11)

where
In order to obtain results in a general case we will apply formulae derived for the auxiliary system and renewal-theory approach. It is clear that successive vacation cycles $C_i$ for $i = 1, 2, ..., j$ are independent random variables and for $i = 1, 2, ..., j$ have the same distribution. Denote d.f.s of $C_0$ and $C_i$ ($i \geq 1$) by $B_0(\cdot)$ and $B(\cdot)$ respectively. Similarly, let

$$b_0(\lambda) = \int_0^\infty e^{-\lambda t} dB_0(t), \quad b(\lambda) = \int_0^\infty e^{-\lambda t} dB(t), \quad \lambda > 0.$$ \hspace{1cm} (12)

Let $\Phi(\cdot)$ be the renewal function of the delayed renewal process generated by d.f.s $B_0(\cdot)$ and $B(\cdot)$. We have

$$\int_0^\infty e^{-\lambda t} d\Phi(t) = \frac{b_0(\lambda)}{1 - b(\lambda)}, \quad \lambda > 0.$$ \hspace{1cm} (13)

Of course (see (8))

$$b_0(\lambda) = e^{-\lambda C_0} = E_{\text{std}}[e^{-\lambda \tau_1}] = 1 - f_+(1, \lambda, 0).$$ \hspace{1cm} (14)

Now we will find the formula for $b(\lambda)$. We get

$$b(\lambda) = E\{e^{-\lambda C_1}\} = \sum_{j=1}^\infty E\{e^{-\lambda C_1} | r_1 = j\} P\{r_1 = j\}$$

$$= \sum_{j=1}^\infty \int_0^\infty dF_1(y) \sum_{k=1}^\infty \sum_{l=k}^\infty b_l^{(k)} E_l\{e^{-\lambda \tau_1}\} \times$$

$$\times \left[ F_1^{(k)}(z-y) - F_1^{(k+1)}(z-y) \right] G^{(l)}(z),$$ \hspace{1cm} (15)

where $r_1$ and $E_l\{e^{-\lambda \tau_1}\}$ were defined in (2) and (9) respectively. Let $t_0 = 0, t_1, t_2, ...$ be moments at which successive vacation cycles $C_0, C_1, C_2, ...$ respectively begin. Introduce the following notation:

$$q_0(n) = P\{h(t_k) = n\}, \quad q(n) = P\{h(t_k+1) - h(t_k) = n\}, \quad k \geq 1.$$ \hspace{1cm} (16)

Next let us put

$$q_0(\theta) = \sum_{n=1}^\infty q_0(n) \theta^n, \quad q(\theta) = \sum_{n=1}^\infty q(n) \theta^n, \quad |\theta| \leq 1.$$ \hspace{1cm} (17)

Since

$$P\{h(t) = m\} = \sum_{k=0}^\infty P\{h(t) = m, t \in C_k\},$$ \hspace{1cm} (18)

then, firstly

$$P\{h(t) = m, t \in C_0\} = E_{\text{std}}[\hat{h}(t) = m, t \in \tau_1]\}$$ \hspace{1cm} (19)

and, secondly, since successive vacation cycles $C_0, C_1, C_2, ...$ form a delayed renewal process, we get for $k \geq 1$

$$P\{h(t) = m, t \in C_k\} = \sum_{n=k}^m (q_0 * q^{(k-1)*})(n) \times$$

$$\times \int_0^\infty P_0\{h(t-y) = m, t-y \in C_1\} d(\hat{B}_0 + B^{(k-1)*})(y).$$ \hspace{1cm} (20)

where $m \geq k$ and $(q_0 * q^{(k-1)*})(n) = \sum_{i=1}^n q_0(n) q^{(k-1)*}(n-i)$ with the assumption $q^{0*}(0) = 1$. Applying properties of transforms we now get for $|\theta| \leq 1$ and $\lambda > 0$:

$$\sum_{n=0}^\infty \theta^n \int_0^\infty e^{-\lambda t} P\{h(t) = m, t \in C_k\} dt = D_0(\theta, \lambda) b_0(\lambda) b^{(k-1)}(\lambda) q_0(\theta) q^{(k-1)}(\theta).$$ \hspace{1cm} (21)

Finally, taking into consideration formulae (18), (19) and (21) we can state the following theorem that gives the double transform of the probability function of departure process in the system with multiple vacations.
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**Theorem 2.** For any \( \lambda > 0 \) and \( |\theta| < 1 \) the following formula is true

\[
\sum_{m=0}^{\infty} \theta^m \int_{0}^{\infty} e^{-\lambda t} P\{b(t) = m\} dt = \hat{D}_{\text{std}}(\theta, \lambda) + \frac{D_0(\theta, \lambda) b_0(\lambda) g_0(\theta)}{1 - b(\lambda) g(\theta)},
\]

where \( \hat{D}_{\text{std}}(\theta, \lambda) \) and \( D_0(\theta, \lambda) \) are defined in (7) and (11) respectively.

3. Conclusion

Using the formula of total probability and the renewal theory it is possible to direct the investigation of the batch arrival system with multiple vacations to the investigation of the corresponding system without vacations. The representation for departure process was obtained without any assumptions about distributions of random variables characterizing the arrival and service processes and the length of vacation periods. The formula can be useful numerically for more simple systems, since it was written down by means of only “input” distributions of the system and components of a certain factorization identity of Wiener-Hopf type connected with these distributions.

References


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