A SMOOTHING ALGORITHM FOR THE INTENSITY OF A DOUBLY STOCHASTIC POISSON PROCESS

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Abstract. A recursive algorithm is obtained for the fixed-point smoothing estimator of the intensity process of a doubly stochastic Poisson process. The proposed methodology can be applied under the single assumption that the covariance function of the intensity process to be estimated is finite-dimensional. Thus, the provided solution becomes an alternative approach to the traditional estimation algorithms with more restrictive hypotheses.

Keywords: Doubly stochastic Poisson process, finite-dimensional covariance.

1. Introduction

Doubly stochastic Poisson processes (DSPP) were introduced by Cox (1955) as a very general Poisson process whose rate or intensity takes on a stochastic nature of its own. As a consequence, two kinds of randomness are operative in a DSPP: one source of randomness arises from the stochastic rate while another comes from the intrinsic Poisson events. The interest on these processes lies in their applicability in such diverse research fields as modelling communication systems, network theory, nuclear medicine, quantitative financial, among others (see, e.g., Snyder and Miller (1991), Slimane and Le-Ngoc (1995), Laukaitis and Racksuskas (2002), and Bouzas et al. (2006)).

In characterizing these processes, the intensity process is an important parameter and its estimation based on the available observations of the DSPP can be required for determining its counting and time statistics. Specifically, we focus our attention in finding linear fixed-point smoothing estimators which are the best in that they minimize the mean-square error.

It is well known that under this error criterion and the linearity constraint, the solution to this estimation problem is completely determined from the solution to a linear equation which only involves the knowledge of the covariance function of the intensity process. However, from the practical point of view, the question of the efficient computation of such an estimate must be solved satisfactorily. In this sense, several approaches which lead to recursive algorithms for the computation of the linear least mean-square error (LLMSE) estimator for the intensity process of an observed DSPP can be found in the literature (Snyder and Miller, 1991).

The most popular of all these algorithms is the Kalman filter which can be applied under the hypothesis that the intensity process to be estimated satisfies a state-space model. Alternatively, in those situations where the intensity process does not admit such a model, it is possible to design efficient algorithms for the LLMSE estimator of the intensity process under less restrictive hypotheses. In this framework, recursive procedures for the computation of the LLMSE filter and predictor of the intensity of a DSPP are devised in Fernández-Alcalá et al. (2008), under the single hypothesis that the covariance function of the intensity process is finite-dimensional.

In this paper, we follow this last approach in order to provide an efficient algorithm for the LLMSE fixed-point smoothing estimate of the intensity of a DSPP, based on its counting observations. Specifically, this estimation problem is formulated in Section 2 and the formulas of the proposed LLMSE fixed-point smoothing algorithm is devised in Section 3.
2. Problem formulation

Let \( \{N(t), t \geq t_0\} \) be a DSPP whose intensity \( \{\lambda(t), t \geq t_0\} \) is a non-negative stochastic process with the first and second-order statistics known and whose covariance function \( R_\lambda(t,s) \) is finite-dimensional, i.e., it can be expressed in the form

\[
R_\lambda(t,s) = \begin{cases} 
  a'(t)b(s), & t \geq s \\
  b'(t)a(s), & t \leq s.
\end{cases}
\]  

where \( a(\cdot) \) and \( b(\cdot) \) are vector-valued functions of dimension \( q \). Let \( E[\lambda(t)] \) denote the mean function of the intensity process \( \{\lambda(t), t \geq t_0\} \).

This process is assumed to be observed in an interval \([t_0,t_f]\) which is partitioned into \( m \) disjoint intervals according to the instants of times \( t_0 < t_1 < t_2 < \ldots < t_m = t_f \), and the number of points occurring in each subinterval is considered. Let \( \{N_1,N_2,\ldots,N_m\} \), with \( N_i = N(t_i) - N(t_{i-1}) \), denote these counting observations whose mean function \( E[N_i] \) and covariance function \( R_N(t_i,t_j) \) are given by the expressions

\[
E[N_i] = \int_{t_{i-1}}^{t_i} E[\lambda(\sigma)]d\sigma, \quad R_N(t_i,t_j) = \int_{t_{j-1}}^{t_j} \int_{t_{i-1}}^{t_i} R_\lambda(\sigma,\tau)d\sigma d\tau + E[N_i]\delta_{ij}
\]  

where \( \delta_{ij} \) is the Kronecker delta function. Moreover, the cross-covariance function between \( \lambda(t) \) and \( N_i \), \( R_{\lambda N}(t_i,t_j) \), is of the form

\[
R_{\lambda N}(t_i,t_j) = \int_{t_{j-1}}^{t_j} R_\lambda(t,\sigma)d\sigma
\]  

Then, based on the set of observations \( \{N_1,N_2,\ldots,N_m\} \), we aim to estimate the intensity process \( \lambda(t) \) at the fixed instant of time \( t = t_p < t_m \). This estimate, named the fixed-point smoother, will be denoted by \( \hat{\lambda}(t_p/t_{m}) \).

Under the linearity constraint and considering the minimization of the mean-square error as optimality criterion, this estimate can be written in terms of the set of counting observations as follows (Snyder, 1991)

\[
\hat{\lambda}(t_p/t_{m}) = E[\lambda(t_p)] + \sum_{i=1}^{m} h(t_p,t_i,t_m)[N_i - E[N_i]], \quad t_p < t_m
\]  

where the optimum impulse-response \( h(t_p,\cdot,t_m) \) that minimizes the mean-square error \( P(t_p/t_{m}) = E[(\lambda(t_p) - \hat{\lambda}(t_p/t_{m}))^2] \) is the solution to the equation

\[
R_{\lambda N}(t_p,t_j) = \sum_{i=1}^{m} h(t_p,t_i,t_m)R_N(t_i,t_j), \quad t_i \leq t_j \leq t_m
\]  

which can be completely determined only from the knowledge of the covariance function of the intensity process \( R_\lambda(t,s) \).

Next, using the fact that \( R_\lambda(t,s) \) is a finite-dimensional covariance function of the form (1), a recursive algorithm for the LLMSE fixed-point smoother (4) is developed in the following section.

3. Fixed-point smoothing algorithm

In the next theorem, an efficient algorithm is provided for the LLMSE fixed-point smoothing estimate \( \hat{\lambda}(t_p/t_{m}) \), \( t_p < t_m \), of the intensity process \( \lambda(t) \) of a DSPP \( N(t) \), based on the discrete time counting observations \( \{N_1,N_2,\ldots,N_m\} \).

**Theorem 1.** The LLMSE fixed-point smoother of the intensity process \( \lambda(t) \), \( \hat{\lambda}(t_p/t_{m}) \) for \( t_p < t_m \), can be obtained from the equation
\[ \hat{\lambda}(t_p / t_m) = \lambda(t_p / t_{m-1}) + h(t_p, t_m, t_m) [N_m - E[N_m] - a'(t_m)e(t_{m-1})] \] (6)

with the initial condition at \( t_m = t_p \), the filter of \( \lambda(t) \)

\[ \hat{\lambda}(t_p / t_p) = E[\lambda(t_p)] + a'(t_p)e(t_p) \] (7)

The \( q \)-dimensional vector \( e(t_m) \) is recursively computed as follows

\[ e(t_m) = e(t_{m-1}) + g(t_m, t_m) [N_m - E[N_m] - a'(t_m)e(t_{m-1})] \]

\[ e(t_0) = 0_q \] (8)

with \( a(t_m) = \int_{t_{m-1}}^{t_m} a(s)ds \), \( 0_q \) the \( q \)-dimensional vector whose elements are all zero and the function \( g(t_m, t_m) \) of the form

\[ g(t_m, t_m) = [\beta(t_m) - Q(t_m-1)a(t_m)]\pi(t_m)^{-1} \] (9)

where \( \beta(t_m) = \int_{t_{m-1}}^{t_m} b(s)ds \), \( \pi(t_m) = [R_N(t_m, t_m) - a'(t_m)Q(t_m-1)a(t_m)] \) and the \( q \times q \)-dimensional matrix \( Q(t_m) \) satisfies the recursive equation

\[ Q(t_m) = Q(t_{m-1}) + g(t_m, t_m)\beta'(t_m) - a'(t_m)Q(t_{m-1}) \]

\[ Q(t_0) = 0_{q \times q} \] (10)

with \( 0_{q \times q} \) the \( q \times q \)-dimensional matrix whose elements are all zero.

Moreover,

\[ h(t_p, t_m, t_m) = [\beta'(t_p) - f'(t_p, t_{m-1})]a(t_m)\pi(t_m)^{-1} \] (11)

with \( f'(t_p, t_m) \) satisfying the expression

\[ f'(t_p, t_m) = f'(t_p, t_{m-1}) + h(t_p, t_m, t_m)\beta'(t_m) - a'(t_m)Q(t_{m-1}) \]

\[ f'(t_p, t_p) = a'(t_p)Q(t_p) \] (12)

**Proof**

Firstly, we give a recursive expression for the impulse response function \( h(t_p, t_j, t_m) \). For that, (2) and (3) are used to rewrite (5) as

\[ h(t_p, t_j, t_m) = \sum_{i=1}^{m} h(t_p, t_i, t_m) \int_{t_{i-1}}^{t_i} R_N(t, \tau) d\tau \] (13)

Then, taking the above equation for \( h(t_p, t_j, t_m) \) and \( h(t_p, t_j, t_{m-1}) \) and taking (1) into account, we obtain that

\[ h(t_p, t_j, t_m) - h(t_p, t_j, t_{m-1}) = -h(t_p, t_m, t_m)a'(t_m)g(t_j, t_{m-1}) \] (14)

with the function \( g(t_j, t_m) \) satisfying the equation, for \( t_1 \leq t_j \leq t_m \),

\[ g(t_j, t_m) = \beta(t_j) - \sum_{i=1}^{m} g(t_i, t_m) \int_{t_{i-1}}^{t_i} R_N(t, \tau) d\tau \] (15)

Now, reasoning in a similar way on the equation (15), we have

\[ g(t_j, t_m) - g(t_j, t_{m-1}) = -g(t_m, t_m)a'(t_m)g(t_j, t_{m-1}) \] (16)

Moreover, using (1) and (2), the equation (15) for \( t_j = t_m \) becomes

\[ g(t_m, t_m) = \beta(t_m) - \sum_{i=1}^{m-1} g(t_i, t_m)\beta'(t_i)a(t_m) \]

Then, equation (9) follows from (16) and the definition of the functions

\[ Q(t_m) = \sum_{i=1}^{m} g(t_i, t_m)\beta'(t_i) \] (17)

and
\[ \pi(t_m) = \{ R_N(t_m, t_m) - a^*(t_m) Q(t_{m-1}) a(t_m) \} \]  

(18)

Besides, from (16) and (17), equation (10) is derived.

Applying now (1) and (2) in equation (13) for \( t_j = t_m \), it follows that

\[ h(t_p, t_m, t_m) R_N(t_m, t_m) = b(t_p) a(t_m) - \sum_{i=1}^{m-1} h(t_p, t_i, t_m) b'(t_i) a(t_m) \]

Therefore, using (14) in this last equation and the definition of the functions \( Q(t_m) \) and \( \pi(t_m) \) given by (17) and (18) in the resulting expression, we have

\[ h(t_p, t_m, t_m) \pi(t_k) = b'(t_p) a(t_m) - \sum_{i=1}^{m-1} h(t_p, t_i, t_m) b'(t_i) a(t_m) \]

and thus, equation (11) is obtained from the introduction of the function

\[ f'(t_p, t_m) = \sum_{i=1}^{m} h(t_p, t_i, t_m) b'(t_i) \]

(19)

Furthermore, from (14), (17) and (19) it can be check that

\[ f'(t_p, t_p) = f'(t_p, t_{m-1}) + h(t_p, t_{m-1}, t_m) b'(t_{m-1}) - a'(t_m) Q(t_{m-1}) \]

and, for \( t_m = t_p \) (19) becomes

\[ f'(t_p, t_p) = \sum_{i=1}^{p} h(t_p, t_i, t_p) b'(t_i) \]

(20)

Since taking \( t_m = t_p \) in equation (13) and using (1) and (15) in the resulting equation we get that

\[ h(t_p, t_i, t_p) = a'(t_p) g(t_i, t_p) \]

(21)

then (20) can be written in the form \( f'(t_p, t_p) = a'(t_p) \sum_{i=1}^{p} g(t_i, t_p) b'(t_i) \) and hence, (12) is a direct consequence of substituting (17) in this last equation.

Finally, from (4) and (14), and the introduction of the function

\[ e(t_m) = \sum_{i=1}^{m} g(t_i, t_m) \{ N_i - E[N_i] \} \]

(22)

equation (6) is obtained with the initialization at \( t_m = t_p \), the filter of the intensity process \( \hat{\lambda}(t_p / t_p) \).

From (4), (21), and (22), the expression (7) for the filter of \( \lambda(t) \) is deduced. Besides, from (16) and (22) it is not difficult to check that \( e(t_m) \) satisfies the recursive equation (8).

4. Simulation example

To analyze the performance of the proposed algorithm, we consider the on-off modulated light estimation problem treated in Snyder (1991).

In particular, the detector output \{\( N(t), t \geq 0 \)\} here is a DSPP whose intensity process \{\( \lambda(t), t \geq t_0 \)\} has mean and covariance functions \( E[\lambda(t)] = 1.1 \) and \( R_\lambda(t, s) = e^{-|t-s|} \). Then, \( R_\lambda(t, s) \) is a finite-dimensional covariance of the form (1) with \( a(t) = e^{-|t|} \) and \( b(t) = e^{+t} \). We also consider that the photo-detector output is observed during a 10 second interval which is partitioned into 100 disjoint intervals according to the times \( t_i = i/10 \). Thus, on the basis of the observations set \{\( N_1, \ldots, N_{100} \)\}, with \( N_i = N(t_i) - N(t_{i-1}) \), the LLMSE fixed-point smoothing estimations \( \hat{\lambda}(t_p / t_p) \) for \( t_p = 2 < t_m \) are computed and compared with the simulated values for the intensity process in Fig. 1.
Fig. 1. Simulated values for $\lambda(t)$ (solid line) and the LLMSE fixed-point smoothing estimates $\hat{\lambda}(2/t_m)$, for $t_m \geq 2$ (dashed line).

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